Flowfield in the Plane of Symmetry below a Delta Wing

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The flowfield in the plane of symmetry of a thin lifting delta wing with supersonic leading edges is examined for wings with apex angles that are comparable to the Mach angle, as well as for the limiting case of a straight leading edge. For these two cases, a simplified treatment of the interaction between the plane expansion wave emanating from the trailing edge and the three-dimensional bow shock is presented. In the region unaffected by the wing tips, the shock decays inversely with distance from the wing.

I. Introduction

THIS paper is concerned with the steady supersonic flowfield of a supersonic leading edge delta wing at large distances below the wing. The use of the area rule to calculate the far field of the supersonic flow past a delta wing has been questioned by Oswatitsch and Sun.[1] They correctly point out that the plane expansion wave emanating from the trailing edge cannot be accounted for by the area rule or the related Whitham-Walkden theory, which deals only with cylindrical waves. Consequently, they contend that the flow past a delta wing will not resemble that past the equivalent body of revolution in the far field. In fact, they conclude that, because of the bow shock-expansion fan interaction, the strength of the bow shock in the symmetry plane vanishes at a finite distance from the wing. Our treatment shows that in this region the shock strength decays inversely with the distance from the wing. Our calculation considers only the interaction of the shock wave and expansion fan emanating from the trailing edge. Hendriks[2] has shown how the tip cones eventually catch up to the front shock. He does this by numerically calculating the first-order dependence domains for points immediately following the shock in the shock-expansion fan region. Although Hendriks uses the flowfield calculated by Oswatitsch and Sun as the one that the tip cones propagate into, we expect that when the flowfield calculated here is used the final conclusion will be the same: eventually the tip cones bring in additional three-dimensional effects and change the inverse first-power decay found here to the familiar inverse three-quarter power law.

II. Physical Problem

Figure 1 is a sketch of the top and side views of a thin delta wing, with the latter depicting flow in the plane of symmetry. In this view only the waves affecting the bow shock *A-a-b* have been sketched. The flat delta wing, indicated by *A-B* in the side view, is taken to be at an angle of attack ϵ to the freestream which has a Mach number M_∞. The Cartesian coordinate system (x,y,z) is aligned with the freestream direction, and the coordinates x,y,z have been nondimensionalized by the wing length L. We are considering supersonic leading edges, which require $\beta_\infty \tan\phi \equiv \bar{\sigma} > 1$, where ϕ is half the wing apex angle and $\beta_\infty \equiv (M_\infty^2 - 1)^{1/2}$. The region *A-a-B-A* is a region of conical flow. The expansion fan occupies the region bounded by the lines *B-a-b* and *B-a′-b′*. We shall refer to the point a with the subscript a, the shock with the subscript s, and characteristics with the subscript c.

In Fig. 2 the rear views for the two sections, $x<1$ and $x>1$, are sketched. The linear or zeroth-order theory is represented by dashed lines, and the first-order shocks are indicated by solid lines. The wing is the line *P-U-T-S* in Fig. 2a and *P-M-N-S* in Fig. 2b. We first discuss the wave patterns on the basis of linear theory. For $x<1$ the first compressive wave *P-Q-R-S* consists of the plane waves *P-Q* and *S-R*, which, in the rear view, make an angle of $\psi \equiv \sin^{-1}(1/\bar{\sigma})$ with the wing, and an arc of the Mach cone which has its apex at the nose of the wing. For $x>1$ the end of the wing is signaled first by the wave system *P′-E-F-S′*, consisting of the expansion fan *E-F* and the tip cones, segments of which are indicated by *P′-E-M* and *S′-F-N*. The first compressive wave *P′-Q′-R′-S′* consists of the planar waves *P′-Q′* and *S′-R′* and an arc of the Mach cone indicated by *U′-Q′-R′T′*. For bounded $\bar{\sigma}$ the radius of curvature of the Mach cone in the plane of symmetry is exactly the position y of the Mach cone. In the limiting case $\bar{\sigma} \to \infty$, the first compressive wave is plane.

To first order the front shock surface is located upstream of the corresponding linear compressive waves, as indicated by the solid lines in Fig. 2. Since the flow ahead of the shock is uniform, the position of the shock depends on the nature of the flow immediately downstream of it. Thus the value of R_s, the radius of curvature of the shock surface in the plane of symmetry, will depend on the strength of the shock surface in the planar region, corresponding to *R′-S′* and *Q′-P′* of the linear theory, relative to that near the plane of symmetry. In the region $y_s \le y_a$ the radius of curvature R_s will vary continuously from approximately y_s in the case $2\bar{\sigma}(t_a)^{1/2} \ll 1$ to infinity for the limiting case $\bar{\sigma} \to \infty$. Here $t_a \equiv 1/2\beta_\infty y_a$ is the small parameter related to the inverse first power at the distance at which the first characteristic from the expansion fan intersects the shock wave.

III. Method of Solution

Oswatitsch and Sun[1] have calculated the flow in the plane of symmetry for $y \le y_a$. In the conical region the linear perturbations to the flow quantities are calculated by solving the three-dimensional linear wave equation subject to the linearized boundary conditions for the delta wing. This gives a complete description of the flow in the conical region to lowest order. The front shock and rear expansion fan appear as Mach lines emanating from the leading and trailing edges, and never intersect. The jump across the expansion fan is determined by using the limiting procedure described in Ref. 1. This linear theory is corrected, in the plane of symmetry, to provide the first-order solution by placing the linear values for the dependent variable on corrected characteristics. Since the local slope of a characteristic will depend on the flow properties at that point (given by the linear theory), the characteristic

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shapes will, in general, be curved, and the rear expansion wave will spread and intersect the front shock as indicated in Fig. 1. Because of the conical nature of the flow in the region *A-a-B-A*, the shock is straight and therefore of constant strength. The strength is determined easily, since to first order the shock bisects the angle between characteristics of the freestream and the conical region.

The first-order calculation of the flow in the expansion fan differs from that in the conical region in that there is no linear boundary-value problem to be solved and corrected. In fact, the linear approximation can provide the flow properties only on the first and last wave emanating from the trailing edge. In order to calculate the flowfield in the fan, the parameter $\mu(0 \leq \mu \leq 1)$, constant on a characteristic, is introduced. This labels the characteristics, and the numerical value of μ gives a measure of the relative position of the characteristic in the expansion fan. This determines the flow properties along each characteristic in the expansion fan for $y \leq y_a$. As in the conical region, this is used to determine the distortion of each characteristic. The calculations of the conical region, front shock, and expansion fan, which may be found in Ref. 1, are reproduced in Sec. IV with slight changes in notation.

At $y=y_a$ the last wave of the conical region intersects the front shock. For $y \geq y_a$ the expansion fan is bounded on the upstream side by the front shock. Since the shock strength and position are unknown, the flow in the expansion fan can be determined only in terms of the unknown shock strength. In general, any analytical determination of the shock-expansion wave interaction must involve three-dimensional characteristics and the calculation of portions of the flowfield outside the plane of symmetry. For the two special cases $2\bar{\sigma}(t_a)^{1/2} \ll 1$ and $\bar{\sigma} \to \infty$, the position and strength of the front shock may be calculated without a complete description of the flow in the expansion fan. This decoupling of the behavior of the shock and expansion fan in the plane of symmetry allows the calculation to be done solely in the plane of symmetry. For these special cases, we make the assumption that for $y_s \geq y_a$ the expression for radius of curvature R_s of the shock in the plane of symmetry is the same as that for $y \leq y_a$. Thus, for $2\bar{\sigma}(t_a)^{1/2} \ll 1$ we assume that $R_s \approx y_s$ for all $y_s \geq y_a$, and for the limiting case of $\bar{\sigma} \to \infty$ we assume that $R_s \to \infty$ for all $y_s \geq y_a$. Under these assumptions, the flow immediately following the shock will be locally axisymmetric for $2\bar{\sigma}(t_a)^{1/2} \ll 1$ and two-dimensional for the limiting case $\bar{\sigma} \to \infty$ for all y_s, instead of just for $y_s \leq y_a$. The latter limiting case may be recognized as that of a two-dimensional wing. In this case, the preceding assumption is known to be true, because of the symmetry of the problem. In the case $2\bar{\sigma}(t_a)^{1/2} \ll 1$, $R_s \approx y_s$ for some small distance beyond y_a and for very large y_s. Our assumption requires that $R_s \approx y_s$ for all y_s between y_a and infinity.

IV. Calculation of the Flowfield

The flowfield for $y \leq y_a$ has been calculated by Oswatitsch and Sun.[1] These results are presented here with slight changes in notation. The solution to the linear wave equation satisfying the boundary conditions of the wing is found for arbitrary $\bar{\sigma}$. The flow deflection angle θ, which, in the symmetry plane, is proportional to the y component of the linear velocity perturbation, is given by

$$\theta = (2\epsilon/\pi)\tan^{-1} 2\bar{\sigma}(t)^{1/2} \tag{1}$$

where $t \equiv \xi/2\beta_\infty y$, and $\xi \equiv x - \beta_\infty y$ is constant on a linear characteristic. At any point in the symmetry plane the slopes of the characteristics are

$$\mathrm{d}y_c/\mathrm{d}x_c = \tan(\theta + \alpha)$$

where $\alpha \equiv \sin^{-1}(1/M)$, and M is the local Mach number. Expanding this for small θ, we have

$$\mathrm{d}y_c/\mathrm{d}x_c = (1/\beta_\infty)(1 + 2A\beta_\infty\theta)$$

where $A \equiv M_\infty^4(\gamma+1)/4\beta_\infty^4$. Integration with constant ξ gives the general form of the characteristic shapes

$$x_c = \beta_\infty y + \xi - 2A\beta_\infty^2 \int_0^{y_c} \theta(\xi,\eta)\,\mathrm{d}\eta \tag{2}$$

In the conical region, $\theta(\xi,y)$ is given by Eq. (1). Substituting this in Eq. (2) and integrating the characteristic shapes in the conical region, we find

$$x_c = \beta_\infty y + \xi - 4A\beta_\infty^2(\epsilon/\pi) f(y_c,t) \tag{3}$$

where

$$f(y_c,t) = y_c[(1+4\bar{\sigma}^2 t)\tan^{-1} 2\bar{\sigma}(t)^{1/2} - (1+4\bar{\sigma}^2)t\tan^{-1}2\bar{\sigma} + 2\bar{\sigma}(t)^{1/2} - 2\bar{\sigma}t]$$

Because of the conical nature of the flow behind the shock the shock must be straight, which requires $\xi_s/y_s = \text{const}$. Thus the shock strength $s = (\gamma M_\infty^2/\beta_\infty)\theta_s$ is constant. The strength of the shock is determined by requiring that, to first order, the shock bisect the characteristics upstream and downstream of it. This is expressed by

$$\mathrm{d}y_s/\mathrm{d}x_s = 1/\beta_\infty(1 + A\beta_\infty\theta_s)$$

Integrating this for arbitrary $\theta_s = \theta_s(x_s,y_s) = \theta_s(y_s)$, the general equation of the shock shape is

$$x_s = \beta_\infty y_s - \beta_\infty^2 \int_0^{y_s} \theta_s(\eta)\,\mathrm{d}\eta \tag{4}$$

For $y_s \leq y_a$, θ_s is a constant, so that the $y_s \leq y_a$ shock shape is

$$x_s = \beta_\infty[1 - A\beta_\infty\theta_s]y_s \tag{5}$$

Since $t_s = \text{constant}$, Eq. (3) and (5) may be equated at any $y_s \leq y_a$ to give a transcendental equation for t_s and therefore θ_s.

The parameter $\mu(0 \leq \mu \leq 1)$, which is constant on a characteristic, is introduced to calculate the flow in the expansion fan. It gives the relative amount of expansion that the flow experiences up to that characteristic labeled by μ. Thus the flow deflection angle in the $y \leq y_a$ expansion fan is

$$\theta(\mu,y) = \theta(0,y) + \mu\Delta\theta]^{\pm} \tag{6}$$

where $\Delta\theta]^{\pm}$ is the total jump in θ across the expansion fan given by

$$\Delta\theta]\pm = -\frac{2\bar{\sigma}\epsilon\cos^{-1}(1/\bar{\sigma})}{\pi(\bar{\sigma}^2-1)^{1/2}}$$

Setting $\xi = 1$ in Eq. (1) and substituting this in Eq. (6), we find that for $y \leq y_a$ the expansion fan flow is

$$\theta(\mu,y) = \frac{2\epsilon}{\pi}\tan^{-1}\frac{2\bar{\sigma}}{(2\beta_\infty y)^{1/2}} + \mu\Delta\theta]^{\pm} \tag{7}$$

The shapes of the characteristics are obtained by integrating Eq. (2) with $\xi = 1$ and $\theta(\xi,y)$ replaced by $\theta(\mu,y)$

$$x_c = \beta_\infty y_c + 1 - 4A\beta_\infty^2 \frac{\epsilon}{\pi} f(y_c,\ 1/2\beta_\infty y_c), -2A\beta_\infty^2\mu\Delta\theta]^{\pm}$$

These results are essentially those of Ref. 1.

We next derive the shock-expansion fan interaction for the cases $2\bar{\sigma}(t_a)^{1/2} \ll 1$ and $\bar{\sigma} \to \infty$. If we differentiate $\theta = \theta(\mu,y)$ along the shock ($y = y_s$), we have

$$\frac{\mathrm{d}\theta_s}{\mathrm{d}y_s} = \left[\frac{\partial\theta}{\partial\mu}\right]_y \frac{\mathrm{d}\mu_s(y_s)}{\mathrm{d}y_s} + \left[\frac{\partial\theta}{\partial y}\right]_\mu \tag{8}$$

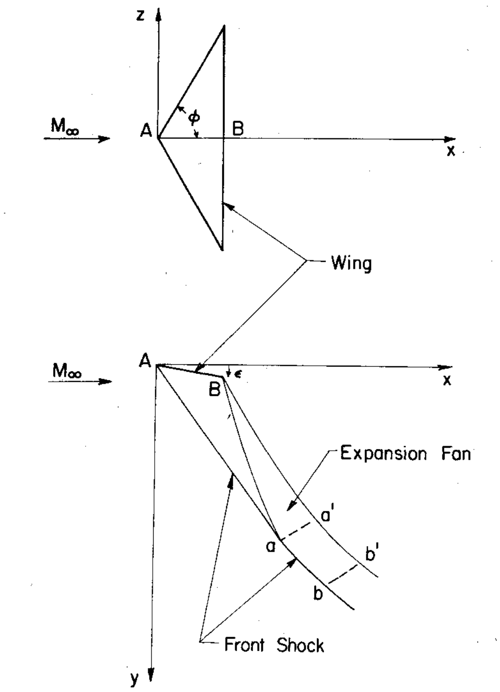

Fig. 1 Top view and section through the plane of symmetry.

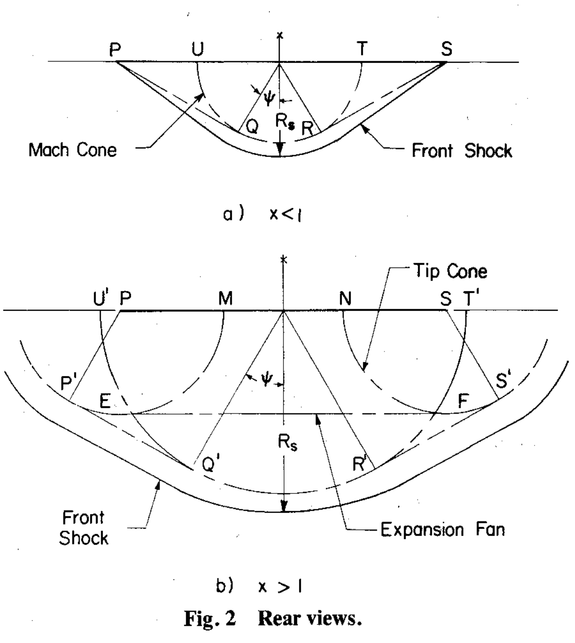

Fig. 2 Rear views.

The first term gives the rate at which the characteristics catch up to the shock. At the shock $x_c = x_s$, so that Eqs. (2) and (4) require

$$1 - 2A\beta_\infty^2 \int_0^{y_s} \theta[\mu_s(y_s),\eta]\,d\eta = -A\beta_\infty^2 \int_0^{y_s} \theta_s(\eta)\,d\eta$$

Differentiating this expression along the shock with respect to y_s and noting that $\theta[\mu_s(y_s),y_s] = \theta_s$, we have

$$\frac{d\mu_s(y_s)}{dy_s} \int_0^{y_s} \left[\frac{\partial\theta}{\partial\mu}\right]_y [\mu_s(y_s),\eta]\,d\eta = -\frac{\theta_s(y_s)}{2}$$

For $(\partial\theta/\partial\mu)_y = \Delta\theta]^{\pm}_{-} = \text{constant}$,

$$\left[\frac{\partial\theta}{\partial\mu}\right]_y \frac{d\mu_s(y_s)}{dy_s} = -\frac{\theta_s(y_s)}{2y_s} \tag{9}$$

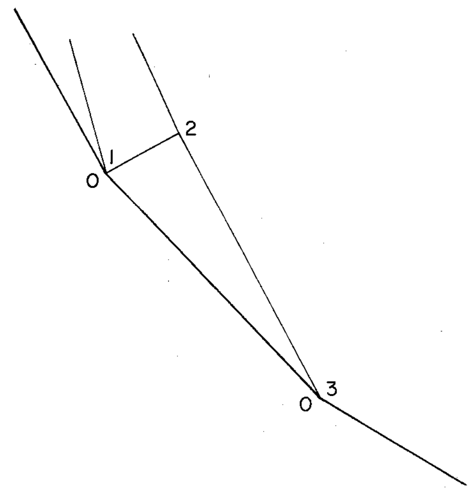

Fig. 3 Shock element.

To determine the second term in Eq. (8), we consider a small section of the shock and the characteristics immediately behind it, as sketched in Fig. 3. The characteristics intersect the shock (1-3) at the points 1 and 3. The line (1-2) is a characteristic of the other family, and the points 0 indicate the uniform freestream conditions. Applying the compatibility conditions along the other family of characteristics, we have

$$\Delta\nu]_0^2 + \Delta\theta]_0^2 = 0 \tag{10}$$

$$\Delta\nu]_0^3 + \Delta\theta]_0^3 = 0 \tag{11}$$

to lowest order, where $d\nu \equiv (M^2-1)^{1/2}(dq/q)$, and q is the magnitude of the velocity vector. Here we have neglected changes in y across the expansion fan compared to changes along it. Equations (10) and (11) hold in the plane of symmetry for all $\bar{\sigma}$. For $2\bar{\sigma}(t_a)^{1/2} \ll 1$, *the flow immediately behind the shock is approximately axisymmetric, and the proper compatibility condition along (2-3) is the axisymmetric one:*

$$\Delta\nu]_2^3 - \Delta\theta]_2^3 = (\theta_2/y_2)\Delta y]_2^3 \tag{12}$$

Eliminating ν from (10-12) and solving for $\Delta\theta]_2^3$, we have

$$\Delta\theta]_2^3 = -(\theta_3/2y_3)\Delta y]_2^3$$

Taking the limit as $\Delta y]_2^3 \to 0$, we obtain the second term in Eq. (8) or $2\bar{\sigma}(t_a)^{1/2} \ll 1$:

$$(\partial\theta/\partial y)_\mu = -\theta_s/2y_s \tag{13}$$

In the case $\bar{\sigma} \to \infty$, the flow immediately behind the shock is two-dimensional, so that the proper compatibility relation along (2-3) is

$$\Delta\nu]_2^3 - \Delta\theta]_2^3 = 0 \tag{14}$$

Again we solve Eqs. (10) and (11) but with Eq. (14) in place of Eq. (12) for $\Delta\theta]_2^3$, and take the limit $\Delta y]_2^3 \to 0$ to obtain the second term of Eq. (8) for $\bar{\sigma} \to \infty$

$$(\partial\theta/\partial y)_\mu = 0 \tag{15}$$

The flow deflection angle θ_s for $2\bar{\sigma}(t_a)^{1/2} \ll 1$ is obtained by substituting Eqs. (9) and (13) in Eq. (8) and integrating with the initial condition of $\theta(y_a) = \theta_a$; the result is

$$\theta_s = \theta_a y_a / y_s \tag{16}$$

Substituting Eqs. (9) and (15) into Eq. (8) and integrating, we have the expression for θ_s when $\sigma \to \infty$:

$$\theta_s = \theta_a (y_a/y_s)^{1/2} \tag{17}$$

The shock shape is calculated by substituting Eqs. (16) and (17) in Eq. (4) and integrating, noting that $\theta_s(\eta) = \theta_a = \text{const}$ for $0 \le \eta \le y_a$. For $2\bar{\sigma}(t_a)^{1/2} \ll 1$, the shock shape for $y \ge y_a$ is

$$x_s = \beta_\infty y_s - A\beta_\infty^2 \theta_a y_a [1 + \ln(y_s/y_a)]$$

For $y \ge y_a$, the shock shape $\bar{\sigma} \to \infty$ is

$$x_s = \beta_\infty y_s + A\beta_\infty^2 \theta_a y_a [1 - 2(y_s/y_a)^{1/2}]$$

The change in θ, as we move away from the shock on a characteristic of the other family, is

$$d\theta = \Delta\theta]^{\pm}_{-} d\mu + (\partial\theta/\partial y)_\mu dy$$

Neglecting changes in y across the fan, compared to changes along it, and integrating, we have

$$\theta(\mu,y) = \theta_s(y) + [\mu - \mu_s(y)]\Delta\theta]^{\pm}_{-}$$

which provides us with the expansion fan flow for $y = y_a$ once $\mu_s(y)$ is determined. For $2\bar{\sigma}(t_a)^{1/2} \ll 1$, we substitute Eq. (16) in Eq. (9) and integrate the result to find

$$\Delta\theta]^{\pm}_{-}\mu_s(y) = \tfrac{1}{2}[\theta_s(y) - \theta_a]$$

where the initial condition $\mu_s(y_a) = 0$ has been used. For $\bar{\sigma} \to \infty$, we substitute Eq. (17) in Eq. (9) and integrate, with the preceding initial condition, to find

$$\Delta\theta]^{\pm}_{-}\mu_s(y) = \theta_s(y) - \theta_a$$

Thus the flow deflection angle in the $y \ge y_a$ expansion fan is

$$\theta(\mu,y) = \tfrac{1}{2}[\theta_s(y) + \theta_a] + \mu\Delta\theta]^{\pm}_{-}$$

for $2\bar{\sigma}(t_a)^{1/2} \ll 1$ and

$$\theta(\mu,y) = \theta_a + \mu\Delta\theta]^{\pm}_{-}$$

for $\bar{\sigma} \to \infty$.

The characteristic shapes are calculated by substituting the expression for $\theta(\mu,y)$ into (2) and integrating, noting that (7) must be used for $0 \le \eta \le y_a$. For $2\bar{\sigma}(t_a)^{1/2} \ll 1$,

$$x_c = \beta_\infty y_c - A\beta_\infty^2 \theta_a y_a \ln(y_c/y_a) - A\beta_\infty^2 \theta_a [1 + (2\Delta\theta/\theta_a)]^{\pm}_{-}\mu] y_c$$

and for $\bar{\sigma} \to \infty$,

$$x_c = \beta_\infty y_c + 1 - 2A\beta_\infty^2 (\theta_a + \mu\Delta\theta]^{\pm}_{-}) y_c$$

V. Conclusion

We have calculated the shock-expansion fan interaction for two cases: $2\bar{\sigma}(t_a)^{1/2} \ll 1$ and $\bar{\sigma} \to \infty$. In the limit $\bar{\sigma} \to \infty$, the shock-expansion fan interaction is that of a two-dimensional wing. For $2\bar{\sigma}(t_a)^{1/2} \ll 1$, the shock strength decays inversely with the distance from the wing, with the proviso that the shock curvature may be approximated by y_s, for all y_s. This means that the expansion fan is continually weakened by its interaction with the shock. In the calculation of Oswatitsch and Sun, the expansion fan is assumed unaffected by the interaction with the shock. Thus, in their calculation the expansion fan remains strong enough to cancel the shock at a finite distance from the wing; this we believe to be incorrect. Within the assumptions made here, it has not been possible to address the question of the transition from an inverse first-power decay to the usual inverse three-quarters power decay, which is associated with the interaction of the wing tip cones with the symmetry plane flow.[2]

Reference

[1] Oswatitsch, K. and Sun, Y.C., "The Wave Formation and Sonic Boom Due to a Delta Wing," *Aeronautical Quarterly*, Vol. 23, May 1972, pp. 87-108.

[2] Hendricks, Th. P.M., "Sonic Boom Theory and the Influence of Three-dimensional Effects," (to be published). Preprint dated March, 1975.